A Brief Introduction to Finite Element Methods for Fluid Flow Problems: Lecture Notes

James Percival

November 11, 2015

AMCG, Department of Earth Science and Engineering
Imperial College London

1 These Notes

These notes accompany the similarly named lecture in the training course for *Fluidity*, the finite element/control volume Navier Stokes solver developed by the Applied Modelling and Computation Group in the Department of Earth Science of Imperial College London. To attend the course please contact Dr. Matthew Piggott, m.piggott@imperial.ac.uk.

Prerequisites:

1. Some basic knowledge of vector calculus.
2. Some basic knowledge of groups, sets & fields.

Key learning objectives:

1. How to write down a weak form of a partial differential equation.
2. Understand what is meant by the order of a Lagrangian polynomial finite element method.

2 Numerical Solution of PDEs

This section revises some basic methods to discretize and solve partial differential equations (PDEs) numerically.

2.1 Introduction

Mathematical descriptions of the equations governing physical processes are often in the form of partial differential equations (PDEs). Variable values are functions of time and satisfy relationships between variables and their partial derivatives. Some architypal examples :

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \qquad \text{(heat equation)}$$

$$\frac{\partial^2 a}{\partial t^2} = c^2 \frac{\partial^2 a}{\partial x^2} \qquad \text{(wave equation)}$$

$$\frac{\partial \tau}{\partial t} + u\frac{\partial \tau}{\partial x} = \kappa \frac{\partial^2 \tau}{\partial^2 x} \qquad \text{(advection-diffusion equation)}$$

Frequently we would like to generate numerical solutions to these equation sets using a computer. However computers (and humans) possess only a finite amount of memory to store the results of calculations, while even as simple a problem domain as the real numbers between 0 and 1 provides an uncountably infinite number of locations for which data would be required. As such, it is necessary to discretize the problem down to operations on a finite dimensional space on which is tractable to perform calculations. There are a number of approaches aimed at achieving this. In this section we will discuss several approaches you may already be familiar with, then introduce the finite element method, which you may not.

2.2 Standard Numerical Methods for PDEs

2.2.1 Finite Difference Methods

Finite difference methods are often considered one of the conseptually simplest forms of numerical discretization, perhaps because they are one of the first methods introduced in most courses on the subject. The underlying assumptions have much in common with other methods, namely that a limit method used in developing the exact statement of a problem is assumed to hold true even when the limiting quantity has not vanished.

In this case, we start from the statement that a partial derivative may be defined as the limit of the ratio of finite differences of the values of a function and of the relevant independent variable,

$$\frac{\partial f}{\partial x_n} = \lim_{h \to 0} \frac{f\left(\boldsymbol{x} + h\hat{\boldsymbol{x}}_n\right) - f\left(\boldsymbol{x}\right)}{\|\left(\boldsymbol{x} + h\hat{\boldsymbol{x}}_n\right) - \left(\boldsymbol{x}\right)\|}.$$

The finite difference method then reverses this relation,

$$\frac{f\left(\boldsymbol{x} + h\hat{\boldsymbol{x}}_n\right) - f\left(\boldsymbol{x}\right)}{\|\left(\boldsymbol{x} + h\hat{\boldsymbol{x}}_n\right) - \left(\boldsymbol{x}\right)\|} = \frac{\partial f}{\partial x_n} + \mathcal{O}\left(h\right) \approx \frac{\partial f}{\partial x_n}.$$

Higher order order derivatives are dealt with by interating this process, for example the famous second order stencil on uniform meshes,

$$\frac{d^2 f}{dx^2} \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + \mathcal{O}\left(h^2\right).$$

More complicated stencils can be found using Taylor series expansions,

$$f(x+h) = f(x) + h\frac{df}{dx} + \frac{h^2}{2}\frac{d^2 f}{dx^2} + \ldots = \sum_{i=0}^{\infty} \frac{h^i}{i!}\frac{d^{(i)} f}{dx^i},$$

and assuming (hoping) that the function remains sufficiently well behaved.

2.2.2 Finite Volume Methods

Consider the classical model of flow through a fluid parcel,

$$\frac{d}{dt}\int_{\text{volume}} a dV = -\sum_{\text{faces}} \int a\boldsymbol{u}\cdot\boldsymbol{n} dS.$$

In the inifinitesimal limit this relation gives the convective equation

$$\frac{\partial a}{\partial t} + \nabla\cdot a\boldsymbol{u} = 0,$$

however finite volume methods use the integral form directly, relating the change of an average value of a function in a finite polyhedral subvolume of a domain to the average fluxes of material across the faces of the polyhedron. Since the finite volume method preserves only the averaged value of the function within the cell, gradients appearing in the fluxes (indeed, in general the face values themselves) may need to be calculated using some other means.

Provided the same face value is used when calculating the amount of material leaving one cell and the amount of material entering a neighbour, then the total quantity of material is conserved, both globally across the entire domain and locally between pairs of cells. These conservation properties are often considered important controls on the accuracy of numerical solution trajectories, particularly for systems with emergent balance properties governing their long term or leading order behaviour.

2.2.3 Spectral Methods

For sufficiently smooth functions over sufficiently simple domains, it is trivial to define a representation by an infinite series of basis functions, each having global support over the entire domain, as well as a suitable basis orthogonality condition. A simple example is the one dimensional Fourier series for a periodic function on an interval,

$$\psi(x) = b_0 + \sum_{n=1}^{\infty} a_n \sin\left(\frac{2\pi n x}{L}\right) + b_n \cos\left(\frac{2\pi n x}{L}\right) \quad x \in [0, L].$$

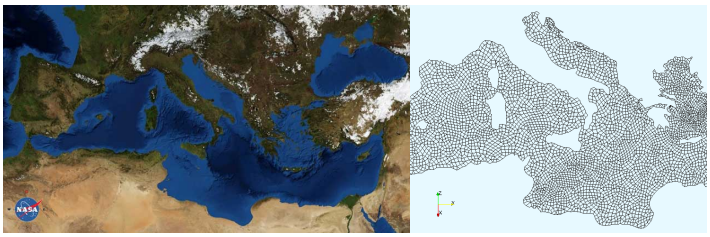

Figure 1: Consider the example of the Mediterranean sea (left) and a two dimensional partition/discretization

The actual values of the coefficients in the equation above can be found using the orthogonality condition for the trignometric functions, i.e

$$\int_0^1 \left(\sin\left(2\pi kx\right) + \cos\left(2\pi mx\right)\right)\left(a\sin\left(2\pi lx\right) + b\cos\left(2\pi nx\right)\right)dx = 2a\delta_{kl} + 2b\delta_{mn},$$

where δ_{ij} is the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise,} \end{cases}$$

Using this approach, a PDE, for example the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

can be recast as an equation for the a_is and b_is,

$$\frac{\partial^2 a_n}{\partial t^2} = -\frac{4\pi^2 n^2 c^2}{L^2} a_n,$$

$$\frac{\partial^2 b_n}{\partial t^2} = -\frac{4\pi^2 n^2 c^2}{L^2} b_n.$$

For a more complicated nonlinear PDE we will generally reduce the problem to an infinite series of coupled ODEs. Discretization is achieved by assuming that most of the energy will remain trapped above a finite wavelength and trunctating the expansions at that level,

$$\tilde{\psi} = b_0 + \sum_{n=1}^{N} a_n \sin\left(\frac{2\pi nx}{L}\right) + b_n \cos\left(\frac{2\pi nx}{L}\right) \approx \psi.$$

The spectral method is particularly suitable for linear elliptic problems, in which case the equations for the coefficients can be solved for independently, and (if the problem is smooth) the convergence to the true solution with N is remarkably fast, the so called "exponential convergence".

2.2.4 Finite Element Methods

The spectral method is of high order and converges quickly, while the finite volume method works well in complicated domains and when the functions being modelled are discontinuous. This begs the question, can we combine the two approaches to get the best of both worlds?

Lets begin by performing a polyhedral domain decomposition, as in the case of the finite volume method.

$$\Omega = \cup_e \Omega^{(e)}$$

Now define a set of local (compact) discrete basis functions, N_i, over the polyhedra, and solve a suitable weak (integral) form of the original PDE over functions in the space of this class of functions. As the restrictions on the function space are successively relaxed, either by reducing the size of the polyhedra, or by increasing the number of basis functions in each polyhedron, it is to be hoped that the finite element solution will converge to the exact one.

2.3 Hybrid Methods

It is common for discretization methods to be combined, especially when solving large coupled problems which have parts which possess properties well suited for different individual methods. For example, finite volume methods are often used for tracer transport equations, where their local and global conservation properties are extremely useful for maintaining balanced trajectories over long time integrations. Similarly, the finite volume method can be coupled to a finite element method to produce a so called "control volume" method.

2.4 Other Approaches

It shouldn't be assumed that the methods presented here form an exhaustive description of all possible numerical methods used to solve PDEs. For example mesh free methods such as smoothed particle hydrodyanmics (SPH) and the discrete element method (DEM)simulate flow problems by introducing physical particles possessing mass and density, which move with the flow and carry information around with them. For SPH the properties of the fluid continuum can then be calculated by averaging the values of the fluid particles, while in DEM the (solid) particles may experience collisions, accreation, breakup or other physical processes.

3 Finite Elements from First Principles

In this section we will construct a matrix equation for a simple linear discretization of the one dimensional Poisson's equation.

3.1 Our Example: Poisson's Equation

We will attempt to keep a concrete exambpl in mind over the course of this section, namely Poisson's equation,

$$\nabla^2 \psi = f(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \Omega. \qquad \text{(Poisson equation)}$$

We will further assume that the problem domain possesses sufficient boundary conditions to guarantee the well-posedness of the solution, namely that the boundary of the domain, $\delta\Omega$ can be partititioned (without remainder into two disjoint subspaces, $\delta\Omega^D$ and $\delta\Omega^N$, ie where

$$\psi(\boldsymbol{x}) = a(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \delta\Omega^D, \qquad \text{(Dirichlet conditions)}$$

$$\boldsymbol{n} \cdot \nabla \psi(\boldsymbol{x}) = b(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \delta\Omega^N \qquad \text{(Neumann conditions)}$$

Specialising to one dimension over the interval $[0, 1]$ a Dirichlet condition on the left and a Neumann condition on the right:

$$\frac{d^2\psi}{dx^2} = f(x) \ \forall x \in [0, 1], \qquad \text{(Poisson equation)}$$

$$\psi(0) = \ \forall \boldsymbol{x} \in \delta\Omega^D, \qquad \text{(Dirichlet conditions)}$$

$$\frac{d\psi}{dx} = b \text{ at } x = 1. \qquad \text{(Neumann conditions)}$$

3.2 Strong & Weak Forms

3.2.1 The Strong Form of a PDE

A partial differential equation expressed pointwise as a relation between the derivatives of a function. This is the form in which partial differential equations have traditionally been written, for example our Poisson equation:

$$\frac{d^2\psi}{dx^2} = f(x) \ \forall x \in [0, 1].$$

By rearranging, this form can be finessed into two other forms,

$$\mathcal{L}\psi = 0 \forall x \in [0, 1]$$

where $\mathcal{L}$ is the differential operator

$$\mathcal{L}[a] = \frac{d^2 a}{dx^2} - f,$$

or the functional form,

$$\mathcal{L}_\psi(x) \forall x \in [0, 1].$$

A potential solution of the operator problem can thus tested by checking that the function vanishes for all x. This implies that any discretization of the strong form of the equations will require discretizing the space for x and, since the operator implicily contains terms in x, discretizing the operatir itlsef. This is fundamentally the recipe for a finite difference method.

3.2.2 The Weak Form of a PDE

This defines an integral functional equation, taking a function as a parameter as well as a function space, the test space, for which the equality is expected to hold. For example,

$$I_{\psi}(\phi) := \int_0^1 \phi \left(\frac{d^2\psi}{dx^2} - f \right) dV = 0, \ \forall \phi \in \mathcal{H}^1[0,1]$$

Here the test space is $\mathcal{H}^1[0,1]$, the set of square integrable functions with square integrable derivatives (see later for more information on function spaces). The function ϕ, called a test function, is an arbitrary member of this space, for which we expect the equation to hold.

Although we are still testing over an infinite number of members of a space, they are now functions taken from a function space rather than points taken from a physical vector space. The existance and uniqueness of solutions depends on the choice of space in which ψ and ϕ live.

3.3 Vector Spaces and Function Spaces.

A set S is a vector space over a field F if there exist operations $+ : S \times S \to S$ (addition) and $\cdot : F \times S \to S$ (scalar multiplication) which satisfy the following eight axioms:

1. Assosiativity of addition,

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \, \forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in S.$$

2. Commutivity of addition,

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \forall \boldsymbol{a}, \boldsymbol{b} \in S.$$

3. Existence of an additive identity,

$$\exists \boldsymbol{0} \in S \text{ such that } \boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}, \forall \boldsymbol{a} \in S.$$

4. Existence of an additive inverse,

$$\forall \boldsymbol{a} \in S, \exists (-\boldsymbol{a}) \in S \text{ such that } \ -\boldsymbol{a} + \boldsymbol{a} = \boldsymbol{0}.$$

5. Multiplicative distributivity over a,

$$\alpha \cdot (\boldsymbol{a} + \boldsymbol{b}) = \alpha \cdot \boldsymbol{a} + \alpha \cdot \boldsymbol{b} \forall (\alpha \in F, \boldsymbol{a}, \boldsymbol{b} \in S).$$

6. Distributitity of scalar addition over scalar multiplication,

$$(\alpha + \beta) \cdot \boldsymbol{a} = \alpha \cdot \boldsymbol{a} + \beta \cdot \boldsymbol{a} \forall (\alpha, \beta \in F, \boldsymbol{a} \in S).$$

7. Associativity of scalar multiplication,

$$\alpha \cdot (\beta \cdot \boldsymbol{a}) = (\alpha \times \beta) \cdot \boldsymbol{a} \forall (\alpha, \beta \in F, \boldsymbol{a} \in S).$$

8. Existence of scalar multiplicative identity,

$$\exists 1 \in F, \text{ such that } 1 \cdot \boldsymbol{a} = \boldsymbol{a} \forall \boldsymbol{a} \in S.$$

This generalises the commonly understood behaviour of vectors, where the addititon and multiplication are assumed to take place componenent by component. Real functions on a domain Ω can be shown to form a function space over the reals when addition and multiplication are defined to act on the function values pointwise, ie.

$$(f+g)(x) = f(x) + g(x),$$
$$(\alpha f)(x) = \alpha \times f(x).$$

This vector space of functions is called a function space.

Vector spaces (and function spaces) may possess subspaces, that is, there may be subsets of the original set for which all the rules given above still apply. As a trivial example of a vector subspace, given the set $\mathbb{R}^3$ of real three-vectors, forming a over the integers $\mathbb{Z}$, we can see that $\mathbb{Z}^3$, the set of integral three vectors is a subset of $\mathbb{R}^3$ and still forms a vector space over the integers. Similarly, given the set, $\mathcal{C}_0[-1,1]$ of continuous functions over the interval $[-1,1]$ we can define a number of subspaces eg:

- Polynomials, $f(x) = 1 + 3x + 4x^2 + 5x^3$
- Continuous functions on a subinterval, $f(x) = \begin{cases} 0 & x < 0, \\ e^x - 1 & 0 \leq x \leq 1, \end{cases}$
- Twice differentiable functions, $f(x) = \begin{cases} x^2 + 2, & x < 0, \\ 2(e^x - x). & x \geq 0. \end{cases}$

Consider the set, S, of continuous functions which are piecewise linear over subintervals $[x_i, x_{i+1}]$, where $x_i = \frac{i}{N}$, for some interval N, This defines a function subspace as before, but now the distance to an arbitrary function in $\mathcal{C}_0[-1,1]$, defined by

$$\langle f, g \rangle = \inf_{g \in S} \int_{-1}^{1} |f - g| \, dx$$

can be shown to be bounded by a function of $1/N$. This implies that this subspace approximates the full function space, and that the approximation improves as the number of intervals increases.

3.4 Boundary Conditions

For a Poisson equation,

$$\nabla^2\psi + f = 0 \forall \boldsymbol{x} \in \Omega$$

to be well posed we require boundary conditions across the whole of $\delta\Omega$ (otherwise the solution will not be unique). Three examples of suitable boundary conditons are:

1. Dirichlet boundary conditions, the value of the function is specified at the boundary: $\psi(\boldsymbol{x}) = a(\boldsymbol{x}) \forall \boldsymbol{x} \delta\Omega^D$.
2. Neumann boundary conditions, the value of the normal derivative of the function is specified on the boundary, $\boldsymbol{n} \cdot \nabla\psi = b(\boldsymbol{x}) \forall \boldsymbol{x} \delta\Omega^N$.
3. Robin boundary conditions, a weighted sum of the function and its normal derivative are specified on the boundary, $\psi + \alpha(\boldsymbol{x}) \boldsymbol{n} \cdot \nabla\psi = c(\boldsymbol{x}) \forall \boldsymbol{x} \delta\Omega^R$.

Under the finite element method, these boundary conditions can either be applied strongly (i.e. enforced pointwise, with subsequent modification of the left and right hand sides of the equation) or weakly (as integral terms appearing explicity in the right hand side of the problem). With elliptic problems it is common for Dirichlet boundary conditions to be applied strongly and Neumann boundary conditions to be applied weakly. This is in some sense the opposite order to the behaviour expected in finite element methods, where Dirichlet conditions appear explicitly and Neumann conditions require techniques such as ghost points.

In the context of our 1D example, we choose to apply a Dirichlet condition, $\psi = 0$ at $x = 0$ and a Neumann condition, $\frac{d\psi}{dx} = 1$ at $x = 1$.

3.4.1 Weak Boundary Conditions

Remember that our original statement of the weak form equation is

$$\int_0^1 \phi \left(\frac{d^2\psi}{dx^2} - f \right) dx = 0.$$

Integrating the differential operator by parts once gives

$$\left[\phi \frac{d\psi}{dx} \right]_0^1 - \int_0^1 \left(\phi f + \frac{d\phi}{fx} \frac{d\psi}{dx} \right) dx = 0.$$

It is this form which we choose to calculate with from now on. Note that the surface term includes the derivative of ψ at $x = 1$, exactly the information provided by our Neumann boundary condition. Substituting we obtain

$$\int_0^1 \frac{d\phi}{fx} \frac{d\psi}{dx} dx + \phi(0) \left. \frac{d\psi}{dx} \right|_{x=0} = \phi(1) + \int_0^1 \phi f \, dx.$$

It now remains to introduce the strong boundary condition.

3.4.2 Strong Boundary Conditions

Suppose ψ is originally from some function space, U. We can define a subspace of this, U_0 where

$$\{f \in U, f(0) = 0\}.$$

We can also choose a function $\psi_d \in U$ with $\psi_d(0) = 1$,. Using the axioms of function spaces this allows us to find $\psi_0 := \psi - \psi_d \in U$ and, from the original definition $\psi_0 \in U_0$. Our new function, ψ_d satisfies a weak equation

$$\int_0^1 \frac{d\phi}{fx}\frac{d\psi_0}{dx}\,dx + \phi(0)\left(\frac{d\psi_0}{dx}\bigg| + \frac{d\psi_d}{dx}\bigg|\right)_{x=0} = -\int_0^1 \frac{d\phi}{fx}\frac{d\psi_d}{dx}\,dx + \phi(1) + \int_0^1 \phi f\,dx.$$

However if the test space is chosen to be U_0 then the unknown surface terms on the left hand side vanish and we have a well posed problem to discretize, namely

$$\int_0^1 \frac{d\phi}{fx}\frac{d\psi_0}{dx}\,dx = -\int_0^1 \frac{d\phi}{fx}\frac{d\psi_d}{dx}\,dx + \phi(1) + \int_0^1 \phi f\,dx.$$

3.5 The Lagrangian Finite element basis

It is now time to define the function space we'll be using. As hinted at before, this will be the space of piecewise linear continuous functions, which we will call $P_1(I_n(0,1))$. Here I is a partition of the unit interval into a finite number of disjoint, connected subintervals,

$$I_n = \{I_1, \ldots, I_n\},$$

$$I_n^{(i)} \cap I_n^{(j)} = 0 \forall j \neq i,$$

with

$$I_N^{(i)} = (x_{i-1}, x_i)$$

for an ordered set of points

$$x_0 = 0, \quad x_{i-1} < x_i, \quad x_n = 1.$$

The closures of these subintervals form a cover of the unit interval

$$\bigcup_{i=1}^{N} cl(I_n^{(i)}) = [0,1],$$

and the intervals will form our elements.

Since our functions are linear, they will, locally look like

$$f_i(x) = a_i + b_i x,$$

while continuity implies

$$a_i + bx_i = a_{i+1}$$

In principle we could store the a coefficient of the leftmost element, and the b values for all elements. In practise, this would tend to produce ill-conditioned matrices unsuitable for numerical solutions. Instead we use the principle of superposition to write

$$\psi(t,x) = \sum_{i=0}^{n} \alpha_i N_i(x),$$

where the N_i are piecewise linear functions which satisfy

$$N_i(x_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Together, these criteria are enough to be able to write down the shape functions direcly,

$$N_i = \begin{cases} 0, & x \leq x_{(i-1)}, \\ \frac{x - x_{(i-1)}}{x_i - x_{(i-1)}}, & x_{(i-1)} < x \leq x_i, \\ \frac{x_{(i+1)} - x}{x_{(i+1)} - x_i}, & x_i < x \leq x_{(i+1)}, \\ 0. & x > x_{(i+1)}. \end{cases}$$

Note that in order to satisfy the boundary condition at $x = 0$, we set $\alpha_0 = 0$ and work with just the shape functions $N_1, \ldots N_n$.

Under the Galerkin approach we use the same basis for the test functions, $\phi = \sum_{i=1}^{n} \beta_i N_i$. Substituting into our exact weak form,

$$\int_0^1 \frac{d\phi}{fx} \frac{d\psi_0}{dx}\, dx = \phi(1) + \int_0^1 \phi f\, dx.$$

we obtain

$$\beta_i \sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} \frac{dN_i}{dx} \frac{dN_j}{dx}\, dx \right] \alpha_j = \beta_n + \beta_i \sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} N_i f\, dx \right].$$

Collecting the coefficents of β_i we have conditions that

$$\sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} \frac{dN_i}{dx} \frac{dN_j}{dx}\, dx \right] \alpha_j = \sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} N_i f\, dx \right],$$

for $i = 1, \ldots, n-1$ and

$$\sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} \frac{dN_n}{dx} \frac{dN_j}{dx}\, dx \right] \alpha_j = 1 + \sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} N_n f\, dx \right],$$

for $i = n$. In matrix form we can express this as

$$\underline{\boldsymbol{D}}\boldsymbol{\alpha} = \boldsymbol{b}$$

where

$$[\underline{\boldsymbol{D}}]_{ij} = \sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} \frac{dN_n}{dx} \frac{dN_j}{dx}\, dx \right]$$

and

$$\boldsymbol{b}_i = \sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} N_i f\, dx \right] + \delta_{in}.$$

3.5.1 The Right Hand Side.

Generally the right hand side of the equation is known explicitly as a function $f : \Omega \to \mathbb{R}$. If this function is known analytically then $\int_0^1 N_i f\, dx$ can usually be calculated exactly. In practice (especially for coupled problems,) it is often simpler to assume it too is represented in an approximated function space,

$$f^{\delta}(x) = \sum_{i=0}^{n_f} \hat{f}_i N_i(x),$$

where (for our choice of shape functions) it is common to take,

$$\hat{f}_i = f(x_i).$$

This gives a matrix form

$$\boldsymbol{b} = \underline{\boldsymbol{M}}\boldsymbol{f}$$

where

$$[\underline{\boldsymbol{M}}]_{ij} = \sum_{e=1}^{n} \left[\int_{x_{e-1}}^{x_e} N_i N_j\, dx \right],$$

and $\boldsymbol{f}$ is the vector of nodal values of f. The matrix $[\underline{\boldsymbol{M}}]_{ij}$ is a common object in finite elements, and is frequently called the mass matrix.

3.6 Some solutions

Suppose $f = 10\sin(5x) + 1/2\cos(3(x + 1/2))$ and we discretize with 6 elements. The matrix equation is then

$$\begin{bmatrix} -\frac{2}{h} & \frac{1}{h} & 0 & 0 & 0 & 0 \\ \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 & 0 & 0 \\ 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 \\ 0 & 0 & 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} \\ 0 & 0 & 0 & 0 & \frac{1}{h} & -\frac{1}{h} \end{bmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix} = \begin{pmatrix} \frac{h}{6}\hat{f}_0 + \frac{2h}{3}\hat{f}_1 + \frac{h}{6}\hat{f}_2 - \frac{1}{h}a \\ \frac{h}{6}\hat{f}_0 + \frac{2h}{3}\hat{f}_2 + \frac{h}{6}\hat{f}_3 \\ \frac{h}{6}\hat{f}_0 + \frac{2h}{3}\hat{f}_3 + \frac{h}{6}\hat{f}_4 \\ \frac{h}{6}\hat{f}_3 + \frac{2h}{3}\hat{f}_4 + \frac{h}{6}\hat{f}_5 \\ \frac{h}{6}\hat{f}_4 + \frac{2h}{3}\hat{f}_5 + \frac{h}{6}\hat{f}_6 \\ \frac{h}{6}\hat{f}_5 + \frac{h}{3}\hat{f}_6 - b \end{pmatrix}.$$

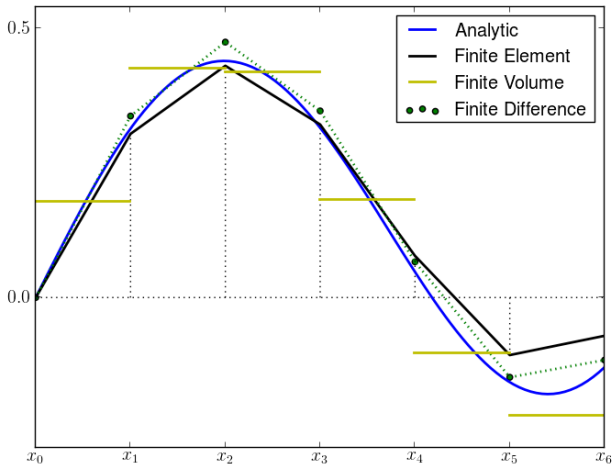

Figure 2: The finite element, finite difference and finite volume solutions to

3.7 Summary

4 Extensions

4.1 Two & Three Dimensional Finite Element Method

The linear one dimensional basis for line elements presented previously generalizes trivially to higher dimensional simplicies (triangles, tetrahedra, etc). These shapes have one more vertex than the space in which they live, and the shape functions can be defined to be unity at one vertex each, and to decay linearly towards each other vertex. Algebraicly, if the simplex has vertices $\boldsymbol{v}_i \in \mathbb{R}^n$, for $i = 1, \ldots, n$ then the jth shape function can be defined as

4.2 High Order Shape Functions

Polynomial orders above linearity are also relatively easy to obtain. In one dimension a basis of order p can be defined by defining $p + 1$ equally spaced points, $x_j^{(e)}$, $j = 0, \ldots, p$ within each element and defining the shape functions to be the pth order polynomials satisfying

$$N_i^{(e)}(x) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

For example in one dimension such second order shape functions are

$$N_0^{(e)} = \frac{(x_2 - x)(x_1 - x)}{(x_2 - x_0)(x_1 - x_0)}$$

$$N_1^{(e)} = \frac{(x_2 - x)(x - x_1)}{(x_2 - x_1)(x_1 - x_0)},$$
$$N_2^{(e)} = \frac{(x - x_1)(x - x_0)}{(x_2 - x_1)(x_2 - x_0)},$$

Note that these shape functions have discontinuities in the first derivative of the solution across element boundaries, these are the so called non-conformal second order elements

4.3 Discontinuous Finite Elements

Another way to increase the number of degrees of freedom is to drop the continuity requirement across element boundaries.

5 Summary

We have now provided a (very) brief overview of the finite element method. You should understand what is meant by the order of a Lagrangian polynomial finite element representation and the difference between continuous versus discontinuous finite element shape functions. The following lectures will now cover some of the higher order thinking necessary to ensure that your numerics remain stable and accurate.

References

[AMCG(2015)] AMCG. *The Fluidity Manual*. Imperial College London, 2015.

[Donea and Huerta(2003)] J. Donea and A. Huerta. *Finite Elements Methods for Flow Problems*. Wiley, 2003.

[Reddy and Gartling(1994)] J. N. Reddy and D. K. Gartling. *The Finite Element Method in Heat Transfer and Fluid Dynamics*. CRC Press, 1994.

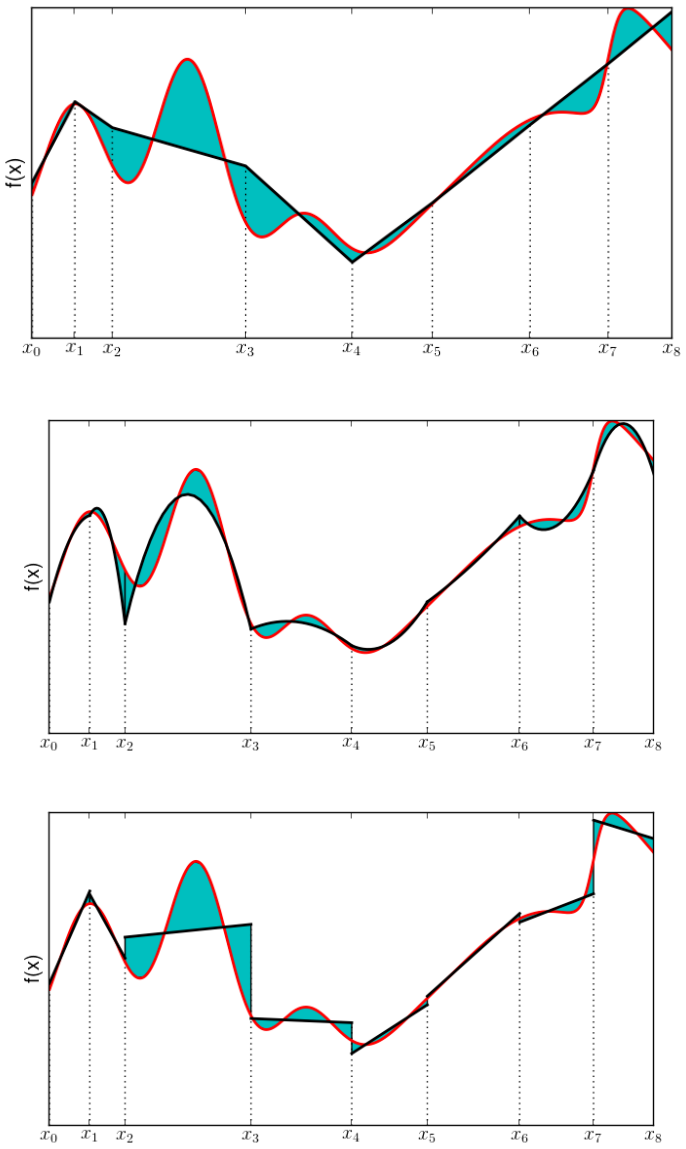

Figure 3: From top to bottom, the figures show the P_1,P_2 and P_1DG Galerkin projections (black) of the function shown in red, on the given line element discretization. The error is coloured in blue. It will be noted that high order functions work well, even for large elements where the function is smooth, whereas discontinuous elements are good at capturing regions with discontinuities or large gradients.